Simplification of Differential Algebraic Equations by the Projection Method[1]

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Outline

Motivation and history

- Goal: reduce higher index DAE to index 1 or ODE
- Method: project dynamic equations onto constraint manifold to systematically eliminate Lagrange multipliers

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Hessenberg form

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$$0 = \mathbf{h}(\mathbf{x}, \mathbf{z})$$

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- Hessenberg index 2: $\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{z}) \\ 0 &= \mathbf{h}(\mathbf{x})\end{aligned}$ $\mathbf{h}_{\mathbf{x}} \cdot \mathbf{f}_{\mathbf{z}}$ nonsingular
- Hessenberg index 3: $\begin{aligned}\dot{\mathbf{y}} &= \mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{z}) \\ \dot{\mathbf{x}} &= \mathbf{g}(\mathbf{x}, \mathbf{y}) \\ 0 &= \mathbf{h}(\mathbf{x})\end{aligned}$ $\mathbf{h}_{\mathbf{x}} \cdot \mathbf{g}_{\mathbf{y}} \cdot \mathbf{f}_{\mathbf{z}}$ nonsingular

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$$\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}, \mathbf{y})$$
$$0 = \mathbf{h}(\mathbf{x})$$

- Hessenberg index m: ...

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Mixed index Hessenberg form

- Hessenberg index 3:

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{z})$$
$$\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}, \mathbf{y}) \qquad \mathbf{h_x} \cdot \mathbf{g_y} \cdot \mathbf{f_z} \text{ nonsingular}$$
$$0 = \mathbf{h}(\mathbf{x})$$

- Hessenberg mixed index 1,3:

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{z}_1, \mathbf{z}_2)$$
$$\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}, \mathbf{y}, \mathbf{z}_1) \qquad \mathbf{h_x} \cdot \mathbf{g_y} \cdot \mathbf{f}_{\mathbf{z}_2} \text{ and } \mathbf{h}_{\mathbf{z}_1} \text{ nonsingular}$$
$$0 = \mathbf{h}(\mathbf{x}, \mathbf{z}_1)$$

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$$0 = \mathbf{h}(\mathbf{x}, \mathbf{z}_1)$$

- x generalized positions
  y generalized velocities
  z algebraic variables (Lagrange multipliers)

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- $\mathbf{x}$ generalized positions
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  $\mathbf{z}$ algebraic variables (Lagrange multipliers)

Classical index reduction

Hessenberg index 3

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{z}) \qquad \#\mathbf{f} = \#\mathbf{y} = n$$

$$\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}, \mathbf{y}) \qquad \#\mathbf{g} = \#\mathbf{x} = n$$

$$0 = \mathbf{h}(\mathbf{x}) \qquad \#\mathbf{h} = \#\mathbf{z} = k \leq n$$

$\mathbf{h}_{\mathbf{x}} \cdot \mathbf{g}_{\mathbf{y}} \cdot \mathbf{f}_{\mathbf{z}}$ nonsingular

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$$0 = \mathbf{h}(\mathbf{x}) \qquad \#\mathbf{h} = \#\mathbf{z} = k \leq n$$

$$0 = \mathbf{h}_{\mathbf{x}} \cdot \dot{\mathbf{x}} \qquad (\mathrm{diff})$$

$\mathbf{h}_{\mathbf{x}} \cdot \mathbf{g}_{\mathbf{y}} \cdot \mathbf{f}_{\mathbf{z}}$ nonsingular

Classical index reduction

Hessenberg index 3

$$
\begin{aligned}
\dot{\mathbf{y}} &= \mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{z}) && \#\mathbf{f} = \#\mathbf{y} = n \\
\dot{\mathbf{x}} &= \mathbf{g}(\mathbf{x}, \mathbf{y}) && \#\mathbf{g} = \#\mathbf{x} = n \\
0 &= \mathbf{h}(\mathbf{x}) && \#\mathbf{h} = \#\mathbf{z} = k \leq n \\
0 &= \mathbf{h}_{\mathbf{x}} \cdot \dot{\mathbf{x}} && \text{(diff)} \\
&= \mathbf{h}_{\mathbf{x}} \cdot \mathbf{g} && \text{(subs)}
\end{aligned}
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$\mathbf{h}_{\mathbf{x}} \cdot \mathbf{g}_{\mathbf{y}} \cdot \mathbf{f}_{\mathbf{z}}$ nonsingular

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&= \mathbf{h}_{\mathbf{x}} \cdot \mathbf{g} && \text{(subs)} \\
0 &= \dot{\mathbf{h}_{\mathbf{x}}} \cdot \mathbf{g} + \mathbf{h}_{\mathbf{x}} \cdot (\mathbf{g}_{\mathbf{x}} \cdot \dot{\mathbf{x}} + \mathbf{g}_{\mathbf{y}} \cdot \dot{\mathbf{y}}) && \text{(diff)}
\end{aligned}
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\begin{aligned}
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0 &= \mathbf{h}(\mathbf{x}) && \#\mathbf{h} = \#\mathbf{z} = k \leq n \\
0 &= \mathbf{h_x} \cdot \dot{\mathbf{x}} && \text{(diff)} \\
&= \mathbf{h_x} \cdot \mathbf{g} && \text{(subs)} \\
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&= \dot{\mathbf{h}}_{\mathbf{x}} \cdot \mathbf{g} + \mathbf{h_x} \cdot \mathbf{g_x} \cdot \mathbf{g} + \mathbf{h_x} \cdot \mathbf{g_y} \cdot \mathbf{f} && \text{(subs)}
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$\mathbf{h_x} \cdot \mathbf{g_y} \cdot \mathbf{f_z}$ nonsingular

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$$\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}, \mathbf{y}) \qquad \#\mathbf{g} = \#\mathbf{x} = n$$

$$0 = \dot{\mathbf{h}_\mathbf{x}} \cdot \mathbf{g} + \mathbf{h}_\mathbf{x} \cdot \mathbf{g}_\mathbf{x} \cdot \mathbf{g} + \mathbf{h}_\mathbf{x} \cdot \mathbf{g}_\mathbf{y} \cdot \mathbf{f}$$

$\mathbf{h}_\mathbf{x} \cdot \mathbf{g}_\mathbf{y} \cdot \mathbf{f}_\mathbf{z}$ nonsingular $\Longrightarrow$ index 1, $\#$DE $= 2n$, $\#$AE $= k$

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$$0 = \dot{\mathbf{h}_{\mathbf{x}}} \cdot \mathbf{g} + \mathbf{h}_{\mathbf{x}} \cdot \mathbf{g}_{\mathbf{x}} \cdot \mathbf{g} + \mathbf{h}_{\mathbf{x}} \cdot \mathbf{g}_{\mathbf{y}} \cdot \mathbf{f}$$

$\mathbf{h}_{\mathbf{x}} \cdot \mathbf{g}_{\mathbf{y}} \cdot \mathbf{f}_{\mathbf{z}}$ nonsingular $\Longrightarrow$ index 1, $\#$DE $= 2n$, $\#$AE $= k$

In the paper: also non-autonomous, $\#\mathbf{x} \neq \#\mathbf{y}$, mixed index

Projection

- diff: $0 = \mathbf{h_x} \cdot \dot{\mathbf{x}} = \mathbf{C} \cdot \dot{\mathbf{x}} \quad \Longrightarrow \quad \dot{\mathbf{x}}$ tangential to $\mathbf{h} = 0$
- Idea: introduce new velocities $\mathbf{u}$ in the tangent space

Projection

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- diff&subs: $\dot{\mathbf{D}} \cdot \mathbf{u} + \mathbf{D} \cdot \dot{\mathbf{u}} = \mathbf{g_x} \cdot \mathbf{g} + \mathbf{g_y} \cdot \mathbf{f}$

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- diff&subs: $\dot{\mathbf{D}} \cdot \mathbf{u} + \mathbf{D} \cdot \dot{\mathbf{u}} = \mathbf{g_x} \cdot \mathbf{g} + \mathbf{g_y} \cdot \mathbf{f}$
- Project onto normal space:

$$\mathbf{C} \cdot \dot{\mathbf{D}} \cdot \mathbf{u} + \mathbf{C} \cdot \mathbf{D} \cdot \dot{\mathbf{u}} = \mathbf{C} \cdot \mathbf{g_x} \cdot \mathbf{g} + \mathbf{C} \cdot \mathbf{g_y} \cdot \mathbf{f}$$

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- diff&subs: $\dot{\mathbf{D}} \cdot \mathbf{u} + \mathbf{D} \cdot \dot{\mathbf{u}} = \mathbf{g_x} \cdot \mathbf{g} + \mathbf{g_y} \cdot \mathbf{f}$
- Project onto normal space: AE for $\mathbf{z}$

$$\mathbf{C} \cdot \dot{\mathbf{D}} \cdot \mathbf{u} = \mathbf{C} \cdot \mathbf{g_x} \cdot \mathbf{g} + \mathbf{C} \cdot \mathbf{g_y} \cdot \mathbf{f}$$

- Project onto tangent space:

$$\mathbf{D}^T \cdot \dot{\mathbf{D}} \cdot \mathbf{u} + \mathbf{D}^T \cdot \mathbf{D} \cdot \dot{\mathbf{u}} = \mathbf{D}^T \cdot \mathbf{g_x} \cdot \mathbf{g} + \mathbf{D}^T \cdot \mathbf{g_y} \cdot \mathbf{f}$$

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- Project onto normal space: AE for $\mathbf{z}$
$$\mathbf{C} \cdot \dot{\mathbf{D}} \cdot \mathbf{u} = \mathbf{C} \cdot \mathbf{g_x} \cdot \mathbf{g} + \mathbf{C} \cdot \mathbf{g_y} \cdot \mathbf{f}$$
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- diff&subs: $\dot{\mathbf{D}} \cdot \mathbf{u} + \mathbf{D} \cdot \dot{\mathbf{u}} = \mathbf{g_x} \cdot \mathbf{g} + \mathbf{g_y} \cdot \mathbf{f}$
- Project onto normal space: AE for $\mathbf{z}$

$$\mathbf{C} \cdot \dot{\mathbf{D}} \cdot \mathbf{u} = \mathbf{C} \cdot \mathbf{g_x} \cdot \mathbf{g} + \mathbf{C} \cdot \mathbf{g_y} \cdot \mathbf{f}$$

- Project onto tangent space: DE for $\mathbf{u}$

$$\mathbf{D}^T \cdot \dot{\mathbf{D}} \cdot \mathbf{u} + \mathbf{D}^T \cdot \mathbf{D} \cdot \dot{\mathbf{u}} = \mathbf{D}^T \cdot \mathbf{g_x} \cdot \mathbf{g} + \mathbf{D}^T \cdot \mathbf{g_y} \cdot \mathbf{f}$$

- DE for $\mathbf{x}$: $\dot{\mathbf{x}} = \mathbf{D} \cdot \mathbf{u}$

Projection

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- Idea: introduce new velocities $\mathbf{u}$ in the tangent space
- $\mathbf{D}$ orthogonal complement: $\mathbf{C} \cdot \mathbf{D} = 0$
- Ansatz: $\dot{\mathbf{x}} = \mathbf{D} \cdot \mathbf{u} = \mathbf{g}, \qquad \# \mathbf{u} = n - k$
- diff&subs: $\dot{\mathbf{D}} \cdot \mathbf{u} + \mathbf{D} \cdot \dot{\mathbf{u}} = \mathbf{g_x} \cdot \mathbf{g} + \mathbf{g_y} \cdot \mathbf{f}$
- Project onto normal space: AE for $\mathbf{z}$

$$\mathbf{C} \cdot \dot{\mathbf{D}} \cdot \mathbf{u} = \mathbf{C} \cdot \mathbf{g_x} \cdot \mathbf{g} + \mathbf{C} \cdot \mathbf{g_y} \cdot \mathbf{f}$$

- Project onto tangent space: DE for $\mathbf{u}$

$$\mathbf{D}^T \cdot \dot{\mathbf{D}} \cdot \mathbf{u} + \mathbf{D}^T \cdot \mathbf{D} \cdot \dot{\mathbf{u}} = \mathbf{D}^T \cdot \mathbf{g_x} \cdot \mathbf{g} + \mathbf{D}^T \cdot \mathbf{g_y} \cdot \mathbf{f}$$

- DE for $\mathbf{x}$: $\dot{\mathbf{x}} = \mathbf{D} \cdot \mathbf{u}$
- AE for $\mathbf{y}$: $\mathbf{D} \cdot \mathbf{u} = \mathbf{g}$

Projection

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$$\mathbf{C} \cdot \dot{\mathbf{D}} \cdot \mathbf{u} = \mathbf{C} \cdot \mathbf{g_x} \cdot \mathbf{g} + \mathbf{C} \cdot \mathbf{g_y} \cdot \mathbf{f}$$

- Project onto tangent space: DE for $\mathbf{u}$

$$\mathbf{D}^T \cdot \dot{\mathbf{D}} \cdot \mathbf{u} + \mathbf{D}^T \cdot \mathbf{D} \cdot \dot{\mathbf{u}} = \mathbf{D}^T \cdot \mathbf{g_x} \cdot \mathbf{g} + \mathbf{D}^T \cdot \mathbf{g_y} \cdot \mathbf{f}$$

- DE for $\mathbf{x}$: $\dot{\mathbf{x}} = \mathbf{D} \cdot \mathbf{u}$
- AE for $\mathbf{y}$: $\mathbf{D} \cdot \mathbf{u} = \mathbf{g}$

$\Longrightarrow$ index 1, $\#$DE $= 2n - k$, $\#$AE $= n + k$

Projection

- diff: $0 = \mathbf{h_x} \cdot \dot{\mathbf{x}} = \mathbf{C} \cdot \dot{\mathbf{x}} \quad \Longrightarrow \quad \dot{\mathbf{x}}$ tangential to $\mathbf{h} = 0$
- Idea: introduce new velocities $\mathbf{u}$ in the tangent space
- $\mathbf{D}$ orthogonal complement: $\mathbf{C} \cdot \mathbf{D} = 0$
- Ansatz: $\dot{\mathbf{x}} = \mathbf{D} \cdot \mathbf{u} = \mathbf{g}, \qquad \#\mathbf{u} = n - k$
- diff&subs: $\dot{\mathbf{D}} \cdot \mathbf{u} + \mathbf{D} \cdot \dot{\mathbf{u}} = \mathbf{g_x} \cdot \mathbf{g} + \mathbf{g_y} \cdot \mathbf{f}$
- Project onto normal space: AE for $\mathbf{z}$
$$\mathbf{C} \cdot \dot{\mathbf{D}} \cdot \mathbf{u} = \mathbf{C} \cdot \mathbf{g_x} \cdot \mathbf{g} + \mathbf{C} \cdot \mathbf{g_y} \cdot \mathbf{f}$$
- Project onto tangent space: DE for $\mathbf{u}$
$$\mathbf{D}^T \cdot \dot{\mathbf{D}} \cdot \mathbf{u} + \mathbf{D}^T \cdot \mathbf{D} \cdot \dot{\mathbf{u}} = \mathbf{D}^T \cdot \mathbf{g_x} \cdot \mathbf{g} + \mathbf{D}^T \cdot \mathbf{g_y} \cdot \mathbf{f}$$
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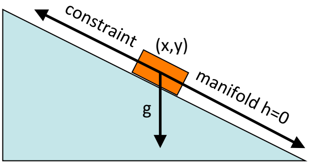
In the paper: also for higher index

Example

Sliding mass on a slope

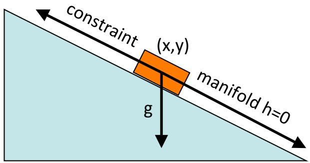


$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} \lambda \\ 2\lambda - g \end{pmatrix}$$

$$0 = x + 2y - 4$$

Example

Sliding mass on a slope



$$\mathbf{y} = \begin{pmatrix} v \\ w \end{pmatrix} \qquad \begin{pmatrix} \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} \lambda \\ 2\lambda - g \end{pmatrix} = \mathbf{f}$$

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} v \\ w \end{pmatrix} = \mathbf{g}$$

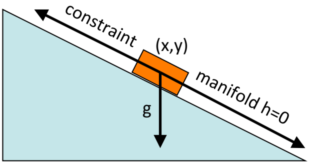
$$\mathbf{z} = \lambda \qquad 0 = x + 2y - 4 = \mathbf{h}$$

Example

Sliding mass on a slope

$\mathbf{C} = (\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}) = (1, 2)$,

$\mathbf{D} = \binom{-2}{1}$



$$\mathbf{y} = \begin{pmatrix} v \\ w \end{pmatrix} \qquad \begin{pmatrix} \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} \lambda \\ 2\lambda - g \end{pmatrix} \qquad = \mathbf{f}$$

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} u = \begin{pmatrix} v \\ w \end{pmatrix} \qquad = \mathbf{g}$$

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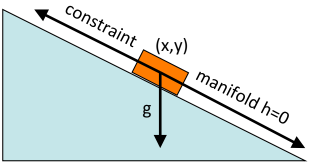

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$$\begin{pmatrix} -2 \\ 1 \end{pmatrix} \dot{u} = \begin{pmatrix} \lambda \\ 2\lambda - g \end{pmatrix} \quad \text{(diff\&subs)}$$

Example

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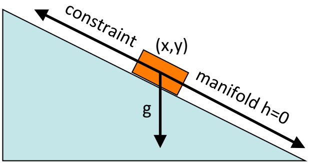

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$$(-2, 1) \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix} \dot{u} = (-2, 1) \cdot \begin{pmatrix} \lambda \\ 2\lambda - g \end{pmatrix} \quad \text{(project)}$$

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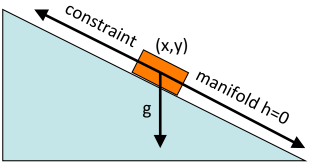

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$$(-2, 1) \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix} \dot{u} = (-2, 1) \cdot \begin{pmatrix} \lambda \\ 2\lambda - g \end{pmatrix} \quad \text{(project)}$$

$$5\dot{u} = 2g \quad \text{(simplify)}$$

Special cases

$$\#\mathbf{x} = \#\mathbf{y} = n, \quad \#\mathbf{h} = k$$

Method	**#DE**	**#AE**
Classical	$2n$	k

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Classical	$2n$	k
Generalized Projection Method	$2n - k$	$k + n$

Special cases

$$\#\mathbf{x} = \#\mathbf{y} = n, \quad \#\mathbf{h} = k$$

Method	#DE	#AE
Classical	$2n$	k
Generalized Projection Method	$2n - k$	$k + n$
(1) $l \leq k$ linear constraints in $\mathbf{h}$	$2n - k - l$	$k + n$

Special cases

$$\#\mathbf{x} = \#\mathbf{y} = n, \quad \#\mathbf{h} = k$$

Method	#DE	#AE
Classical	$2n$	k
Generalized Projection Method	$2n-k$	$k+n$
(1) $l \leq k$ linear constraints in $\mathbf{h}$	$2n-k-l$	$k+n$
(2) $\mathbf{g}$ linear	$2n-k$	k

Special cases

$$\#\mathbf{x} = \#\mathbf{y} = n, \quad \#\mathbf{h} = k$$

Method	#DE	#AE
Classical	$2n$	k
Generalized Projection Method	$2n-k$	$k+n$
(1) $l \leq k$ linear constraints in $\mathbf{h}$	$2n-k-l$	$k+n$
(2) $\mathbf{g}$ linear	$2n-k$	k
(3) $\mathbf{f}$ linear	$2n-k$	n

Special cases

$\#\mathbf{x} = \#\mathbf{y} = n, \quad \#\mathbf{h} = k$

Method	#DE	#AE
Classical	$2n$	k
Generalized Projection Method	$2n-k$	$k+n$
(1) $l \leq k$ linear constraints in $\mathbf{h}$	$2n-k-l$	$k+n$
(2) $\mathbf{g}$ linear	$2n-k$	k
(3) $\mathbf{f}$ linear	$2n-k$	n
(2+3) $\mathbf{f}, \mathbf{g}$ linear	$2n-k$	0

Original Projection Method has (2+3); in fact, $\mathbf{f} = \mathbf{a}(\mathbf{x}, \mathbf{y}) + \mathbf{C}^T \cdot \mathbf{z}$

Benchmarks for some index 3 models

Model	Version	#DE	#AE	#SE	PM Time
DoublePendulum1	HF	22	7	112	
	PM	14	0	129	0.91s
FourBar	HF	30	11	166	
	PM	16	0	194	13.03s
Pendulum1	HF	12	3	58	
	PM	8	0	67	0.29s
SliderCrank	HF	29	10	215	
	PM	18	0	231	2.15s
TriplePendulum1	HF	32	11	165	
	PM	20	0	199	1.86s

Maple implementation

HLMT models converted to (mixed-index) Hessenberg form first (HF)

SE: "solved equations"